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THE PERNICIOUS INFLUENCE OF MATHEMATICS ON SCIENCE

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Our announced subject today is the role of mathematics in the formulation of physical theories. I wish however to make use of the license permitted at philosophical congresses, in two regards: in the first place, to confine myself to the negative aspects of this role, leaving it to others to dwell on the amazing triumphs of the mathematical method; in the second place, to comment not only on physical science but also on social science, in which the characteristic inadequacies which I wish to discuss are more readily apparent.

Computer programmers often make a certain remark about computing machines, which may perhaps be taken as a complaint: that computing machines, with a perfect lack of discrimination, will do any foolish thing they are told to do. The reason for this lies of course in the narrow fixation of the computing machine 'intelligence' upon the basely typographical details of its own perceptions — its inability to be guided by any large context. In a psychological description of the computer intelligence, three related adjectives push themselves forward: single-mindedness, literal-mindedness, simple-mindedness. Recognizing this, we should at the same time recognize that this single-mindedness, literal-mindedness, simple-mindedness also characterizes theoretical mathematics, though to a lesser extent.

It is a continual result of the fact that science tries to deal with reality that even the most precise sciences normally work with more or less ill-understood approximations toward which the scientist must maintain an appropriate skepticism. Thus, for instance, it may come as a shock to the mathematician to learn that the Schrodinger equation for the hydrogen atom, which he is able to solve only after a considerable effort of functional analysis and special function theory, is not a literally correct description of this atom, but only an approximation to a somewhat more correct equation taking account of spin, magnetic dipole, and relativistic effects; that this corrected equation is itself only an ill-understood approximation to an infinite set of quantum field-theoretical equations; and finally that the quantum field theory, besides diverging, neglects a myriad of strange-particle interactions whose strength and form are largely unknown. The physicist, looking at the original Schrodinger equation, learns to sense in it the presence of many invisible terms, integral, integrodifferential, perhaps even more complicated types of operators, in addition to the differential terms visible, and this sense inspires an entirely appropriate disregard for the purely technical features of the equation which he sees. This very healthy self-skepticism is foreign to the mathematical approach.

Mathematics must deal with well-defined situations. Thus, in its relations with science mathematics depends on an intellectual effort outside of mathematics for the crucial specification of the approximation which mathematics is to take literally. Give a mathematician a situation which is the least bit ill-defined — he will first of all make it well defined. Perhaps appropriately, but perhaps also inappropriately. The hydrogen atom illustrates this process nicely. The physicist asks: 'What are the eigenfunctions of such-and-such a differential operator?' The mathematician replies: 'The question as put is not well defined. First you must specify the linear space in which you wish to operate, then the precise domain of the operator as a subspace. Carrying all this out in the simplest way, we find the following result . . .' Whereupon the physicist may answer, much to the mathematician's chagrin: 'Incidentally, I am not so much interested in the operator you have just analyzed as in the following operator, which has four or five additional small terms — how different is the analysis of this modified problem?' In the case just cited, one may perhaps consider that nothing much is lost, nothing at any rate but the vigor and wide sweep of the physicist's less formal attack. But, in other cases, the mathematician's habit of making definite his literal-mindedness may have more unfortunate consequences. The mathematician turns the scientist's theoretical assumptions, i.e., convenient points of analytical emphasis, into axioms, and then takes these axioms literally. This brings with it the danger that he may also persuade the scientist to take these axioms literally. The question, central to the scientific investigation but intensely disturbing in the mathematical context — what happens to all this if the axioms are relaxed? — is thereby put into shadow.

In this way, mathematics has often succeeded in proving, for instance, that the fundamental objects of the scientist's calculations do not exist. The sorry history of the δ-function should teach us the pitfalls of rigor. Used repeatedly by Heaviside in the last century, used constantly and systematically by physicists since the 1920's, this function remained for mathematicians a monstrosity and an amusing example of the physicists' naïveté — until it was realized that the δ-function was not literally a function but a generalized function. It is not hard to surmise that this history will be repeated for many of the notions of mathematical physics which are currently regarded as mathematically questionable. The physicist rightly dreads precise argument, since an argument which is only convincing if precise loses all its force if the assumptions upon which it is based are slightly changed, while an argument which is convincing though imprecise may well be stable under small perturbations of its underlying axioms.

The literal-mindedness of mathematics thus makes it essential, if mathematics is to be appropriately used in science, that the assumptions upon which mathematics is to elaborate be correctly chosen from a larger point of view, invisible to mathematics itself. The single-mindedness of mathematics reinforces this conclusion. Mathematics is able to deal successfully only with the simplest of situations, more precisely, with a complex situation only to the extent that rare good fortune makes this complex situation hinge upon a few dominant simple factors. Beyond the well-traversed path, mathematics loses its bearings in a jungle of unnamed special functions and impenetrable combinatorial particularities. Thus, the mathematical technique can only reach far if it starts from a point close to the simple essentials of a problem which has simple essentials. That form of wisdom which is the opposite of single-mindedness, the ability to keep many threads in hand, to draw for an argument from many disparate sources, is quite foreign to mathematics. This inability accounts for much of the difficulty which mathematics experiences in attempting to penetrate the social sciences. We may perhaps attempt a mathematical economics — but how difficult would be a mathematical history! Mathematics adjusts only with reluctance to the external, and vitally necessary, approximating of the scientists, and shudders each time a batch of small terms is cavalierly erased. Only with difficulty does it find its way to the scientist's ready grasp of the relative importance of many factors. Quite typically, science leaps ahead and mathematics plods behind.

Related to this deficiency of mathematics, and perhaps more productive of rueful consequence, is the simple-mindedness of mathematics — its willingness, like that of a computing machine, to elaborate upon any idea, however absurd; to dress scientific brilliancies and scientific absurdities alike in the impressive uniform of formulae and theorems. Unfortunately however, an absurdity in uniform is far more persuasive than an absurdity unclad. The very fact that a theory appears in mathematical form, that, for instance, a theory has provided the occasion for the application of a fixed-point theorem, or of a result about difference equations, somehow makes us more ready to take it seriously. And the mathematical-intellectual effort of applying the theorem fixes in us the particular point of view of the theory with which we deal, making us blind to whatever appears neither as a dependent nor as an independent parameter in its mathematical formulation. The result, perhaps most common in the social sciences, is bad theory with a mathematical passport. The present point is best established by reference to a few horrible examples. In so large and public a gathering, however, prudence dictates the avoidance of any possible *faux pas*. I confine myself, therefore, to the citation of a delightful passage from Keynes' *General Theory*, in which the issues before us are discussed with a characteristic wisdom and wit:

"It is the great fault of symbolic pseudomathematical methods of formalizing a system of economic analysis . . . that they expressly assume strict independence between the factors involved and lose all their cogency and authority if this hypothesis is disallowed; whereas, in ordinary discourse, where we are not blindly manipulating but know all the time what we are doing and what the words mean, we can keep 'at the back of our heads' the necessary reserves and qualifications and adjustments which we shall have to make later on, in a way in which we cannot keep complicated partial differentials 'at the back' of several pages of algebra which assume they all vanish. Too large a proportion of recent 'mathematical' economics are mere concoctions, as imprecise as the initial assumptions they rest on, which allow the author to lose sight of the complexities and interdependencies of the real world in a maze of pretentious and unhelpful symbols."

The intellectual attractiveness of a mathematical argument, as well as the considerable mental labor involved in following it, makes mathematics a powerful tool of intellectual prestidigitation — a glittering deception in which some are entrapped, and some, alas, entrappers. Thus, for instance, the delicious ingenuity of the Birkhoff ergodic theorem has created the general impression that it must play a central role in the foundations of statistical mechanics.[1] Let us examine this case carefully, and see. Mechanics tells us that the configuration of an isolated system is specified by choice of a point p in its phase surface, and that after t seconds a system initially in the configuration represented by p moves into the configuration represented by $M_t p$. The Birkhoff theorem tells us that if f is any numerical function of the configuration p (and *if* the mechanical system is metrically transitive), the time average

$$\frac{1}{T}\int_0^T f(M_t p)\, dt$$

tends (as $T \to \infty$) to a certain constant; at any rate for all initial configurations p not lying in a set e in the phase surface whose measure $\mu(e)$ is zero; μ here is the (natural) Lebesgue measure in the phase surface. Thus, the familiar argument continues, we should not expect to observe a configuration in which the long-time average of such a function f is not close to its equilibrium value. Here I may conveniently use a bit of mathematical prestidigitation of the very sort to which I object, thus paradoxically making an argument serve the purpose of its own denunciation. Let $\nu(e)$ denote the probability of observing a configuration in the set e; the application of the Birkhoff theorem just made is then justified only if $\mu(e) = 0$ implies that $\nu(e) = 0$. If this is the case, known result of measure theory tells us that $\nu(e)$ is extremely small wherever $\mu(e)$ is extremely small. Now the functions f of principal interest in statistical mechanics are those which, like the local pressure and density of a gas, come into equilibrium, i.e., those functions for which $f(M_t p)$ is constant for long periods of time and for almost all initial configurations p. As is evident by direct computation in simple cases, and as the Birkhoff theorem itself tells us in these cases in which it is applicable, this means that $f(p)$ is close to its equilibrium value except for a set e of configurations of very small measure μ. Thus, not the Birkhoff theorem but the simple and generally unstated hypothesis '$\mu(e) = 0$ implies $\nu(e) = 0$' necessary to make the Birkhoff theorem relevant in any sense at all tells us why we are apt to find $f(p)$ having its equilibrium value. The

[1]This dictum is promulgated, with a characteristically straight face, in Dunford-Schwartz, *Linear Operators*, Vol. I, Chap. 7.

Birkhoff theorem in fact does us the service of establishing its own inability to be more than a questionably relevant superstructure upon this hypothesis.

The phenomenon to be observed here is that of an involved mathematical argument hiding the fact that we understand only poorly what it is based on. This shows, in sophisticated form, the manner in which mathematics, concentrating our attention, makes us blind to its own omissions — what I have already called the single-mindedness of mathematics. Typically, mathematics knows better what to do than why to do it. Probability theory is a famous example. An example which is perhaps of far greater significance is the quantum theory. The mathematical structure of operators in Hilbert space and unitary transformations is clear enough, as are certain features of the interpretation of this mathematics to give physical assertions, particularly assertions about general scattering experiments. But the larger question here, a systematic elaboration of the world-picture which quantum theory provides, is still unanswered. Philosophical questions of the deepest significance may well be involved. Here also, the mathematical formalism may be hiding as much as it reveals.